

# On the Factorization of Network Reliability with Perfect Nodes

Juan Manuel Burgos<sup>a</sup>, Franco Robledo Amoza<sup>b</sup>

<sup>a</sup>*Instituto de Matemáticas, Universidad Nacional Autónoma de México, Unidad Cuernavaca. Av. Universidad s/n, Col. Lomas de Chamilpa. Cuernavaca, Morelos México, 62209.*

*Email: burgos@matcuer.unam.mx*

<sup>b</sup>*Laboratorio de Probabilidad y Estadística  
Facultad de Ingeniería, Universidad de la República  
Julio Herrera y Reissig 565  
11300, Montevideo, Uruguay.*

*Email: frobledo@fing.edu.uy*

---

## Abstract

A new general  $K$ -network reliability factorization theorem is proved. Beside its theoretical mathematical importance the theorem shows how to do parallel processing in exact network reliability calculations in order to reduce the processing time of this NP-hard problem. It also shows a factorization of the Random Cluster Model of Statistical Mechanics in the non cluster limit.

*Keywords:* Network Reliability, Graph Theory, Factorization, Parallel Processing, Random Cluster Model

---

## 1. Introduction

The  $K$ -network reliability factorization theorem [Mo] and the reduction transformations (series-parallel, polygon-to-chain [Wo] and delta-star [Ga]) are the key stone of the known factoring algorithms [SC] for the network reliability exact calculation. Beside these results and the well known factorization through an articulation point, no other general factorization theorem is known in exact  $K$ -network reliability calculation. This paper gives a new general factorization theorem solving the following problem:

**Problem:** *Given a decomposition of a stochastic graph  $G$  by subgraphs  $G_1$  and  $G_2$  only sharing nodes, express the reliability of  $G$  in terms of the*

*reliabilities of the graphs resulting from  $G_1$  and  $G_2$  identifying the common nodes shared by them in all possible ways.*

Exact calculation of  $K$ -network reliability is a NP-hard problem [Co] so it is necessary to know how to do parallel processing in order to reduce the processing time. This theorem has the following immediate parallel processing application: Suppose we are willing to calculate the exact network reliability of a stochastic graph  $G = (V, E, (p_e)_{e \in E})$  with  $m$  edges and perfect nodes. A priori the processing time involve  $2^m$  steps. Cutting the graph  $G$  through a set of nodes obtaining  $G_1$  and  $G_2$  with approximately  $m/2$  edges each, the solution of the above problem allow us to calculate the reliability of  $G$  with  $2^{m/2}$  steps in each processing line.

As another application, it is shown in the appendix a non trivial factorization of the Random Cluster Model of statistical mechanics in the non cluster limit.

## 2. Preliminaries

### 2.1. $K$ -Reliability

The mathematical model of a Network whose nodes are perfect and its edges can fail is a stochastic graph [Co]; i.e. an undirected graph with associated Bernoulli variables to its edges.

**Definition 2.1.** An undirected graph  $G$  is a pair  $(V, E)$  such that  $V$  is finite set whose elements will be called nodes and  $E$  is a subset of  $\{\{a, b\} / a, b \in V\} \times \mathbb{N}$  whose elements will be called edges such that for each pair of distinct edges  $(\{a_1, b_1\}, n_1), (\{a_2, b_2\}, n_2) \in E$  we have that  $n_1 \neq n_2$ .

**Definition 2.2.** A stochastic graph  $G$  is a tern  $(V, E, \Phi)$  such that  $(V, E)$  is a graph and  $\Phi : E \rightarrow Ber$  is a function which associates a Bernoulli variable to each edge in such a way that these variables are independent.

Each Bernoulli variable is characterized by a parameter  $p$  in the  $[0, 1]$  closed interval and we can write a stochastic graph as  $(G, \{p_e\}_{e \in E})$  where  $G$  is an undirected graph and  $p_e$  is the parameter of the variable  $\Phi(e)$ . Nodes and edges of  $G$  will be denoted by  $V(G)$  and  $E(G)$  respectively.

**Definition 2.3.** A state  $\mathcal{E}$  of the graph  $G = (V, E)$  is a function  $\mathcal{E} : E \rightarrow \{0, 1\}$ . An edge  $e$  will be called operative if  $\mathcal{E}(e) = 1$  and will be called non-operative otherwise.

Consider a subset  $K$  of  $V(G)$ . A state  $\mathcal{E}$  of the graph  $G$  will be called a  $K$ -PathSet (or  $K$ -operative) if  $K$  is contained in the set of nodes of any of the edge-connected components of the graph resulting from removing the non-operative edges of  $G$ . Otherwise the state will be called a  $K$ -CutSet.

**Definition 2.4.** The  $K$ -Reliability of a stochastic graph  $G$  is

$$R_K(G) = P(\mathcal{E} \text{ is a } K\text{-PathSet})$$

Because of the independence of the bernoulli variables associated to the edges, we can calculate the  $K$ -Reliability in the following way:

$$P(\mathcal{E}) = \prod_{e_i \in E(G)} p_i^{\mathcal{E}(e_i)} (1 - p_i)^{1 - \mathcal{E}(e_i)} \quad (1)$$

$$R_K(G) = \sum_{\mathcal{E} \text{ is a } K\text{-PathSet}} P(\mathcal{E})$$

## 2.2. Simple Factorization

**Definition 2.5.** Consider an edge  $e = (\{v, w\}, n) \in E$  of a graph  $G = (V, E)$  and define the following equivalence relation in  $V$ :  $a \sim b$  if  $a = b$  or  $\{a, b\} = \{v, w\}$ . Consider the surjective canonical function  $\pi : V \rightarrow V/\sim$  such that  $\pi(a) = [a]_{\sim}$ . We define the contraction of an edge  $e$  in  $G$  as the graph  $G \cdot e$  such that

$$G \cdot e = (V/\sim, E \cdot e)$$

where  $E \cdot e = \{ (\{\pi(a), \pi(b)\}, n) / (\{a, b\}, n) \in E - \{e\} \}$  (see Figure 2). We will denote by  $K_e = \pi(K)$  the new distinguished set of nodes contained in  $V(G \cdot e)$  where  $K$  is the distinguished set of nodes in  $V(G)$ .

**Definition 2.6.** Consider an edge  $e = (\{v, w\}, n) \in E$  of the graph  $G = (V, E)$ . We define the deletion of the edge  $e$  of  $G$  as the graph (see Figure 1)

$$G - e = (V, E - \{e\})$$

The following is the simple factorization proposed for the first time by Moskovitz [Mo]:

**Proposition 2.1.** Consider a stochastic graph  $G$  and a subset  $K$  of its nodes. For any edge  $e_i$  of  $G$  we have that

$$R_K(G) = p_i R_{K_i}(G \cdot e_i) + (1 - p_i) R_K(G - e_i)$$

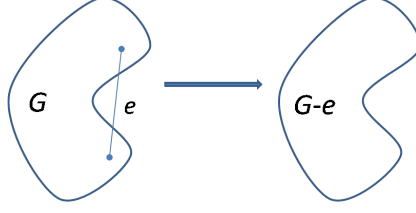


Figure 1: Deletion of the edge  $e$

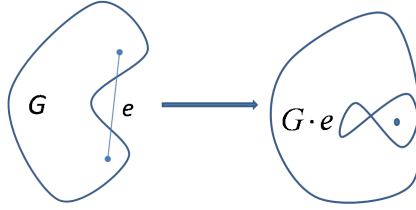


Figure 2: Contraction of the edge  $e$

**Definition 2.7.** Define the following partial order in the set of states of  $G$ :  $\mathcal{E} \leq \mathcal{F}$  if  $\mathcal{E}^{-1}(1) \subset \mathcal{F}^{-1}(1)$ . The state  $\mathcal{E}$  is a  $K$ -minpath if it is minimal in the set of  $K$ -PathSets.

The  $K$ -PathSets have the following coherence property: If  $\mathcal{E} \leq \mathcal{F}$  and  $\mathcal{E}$  is a  $K$ -PathSet, then  $\mathcal{F}$  is a  $K$ -PathSet. For every  $K$ -PathSet  $\mathcal{F}$  there is a  $K$ -minpath  $\mathcal{E}$  such that  $\mathcal{E} \leq \mathcal{F}$ . We conclude that the set of  $K$ -PathSets equals the set of states  $\mathcal{F}$  greater or equal to some  $K$ -minpath. This motivates the following definition [Co]:

**Definition 2.8.** An edge  $e$  of a graph  $G$  will be called  $K$ -irrelevant if  $\mathcal{E}(e) = 0$  for each  $K$ -minpath  $\mathcal{E}$ .

**Proposition 2.2.** If  $e$  is a  $K$ -irrelevant edge of a stochastic graph  $G$ , then

$$R_K(G) = R_K(G - e)$$

### 2.3. $K$ -Reliability Polynomial

Consider a graph  $G$  and the stochastic graph  $G_p$  whose underlying graph is  $G$  and its Bernoulli variables are identical and independent with parameter  $p$ . By proposition 2.1 and the finiteness of the graph, we have that the  $K$ -Reliability is a polynomial respect to the parameter  $p$ :

**Definition 2.9.**  $R_K(p) = R_K(G_p)$  is the es K-Reliability polynomial of  $G$ .

See that

$$R_K(p) = \sum_{i=0}^n C_i^K p^i (1-p)^{n-i}$$

where  $C_i^K$  is the number of  $K$ -pathsets with exactly  $i$  operative edges and  $n$  is the number of edges in  $G$ . In particular, the calculation of the K-Reliability polynomial is equivalent to the calculation of the numbers  $C_i^K$ . Each  $K$ -minpath is a Steiner tree of  $G$  which covers  $K$  (if  $K$  is the set of nodes of  $G$ , then a  $K$ -minpath is a spanning tree). The calculation of the K-Reliability polynomial solves the following *NP*-complete problem: Given a number  $b$  and a subset of nodes  $K$ , Is there a Steiner tree which covers  $K$  with edge number less than or equal to  $b$ ? To answer this question we just have to see whether or not  $m \leq b$  where  $C_m^K$  is the first non zero coefficient of the above expression. In fact, we have the following [Co]:

**Proposition 2.3.** *The Calculation of the K-Reliability polynomial is a NP-hard problem.*

### 3. Combinatorics of the Problem

**Hypothesis 1:** In the whole paper,  $G_1$ ,  $G_2$  and  $G$  are graphs with distinguished subset of nodes  $K_1$ ,  $K_2$  and  $K$  respectively such that  $G = G_1 \cup G_2$ ,  $K = K_1 \cup K_2$  and

$$\{k_1, k_2, \dots, k_n\} = K_1 \cap K_2 = G_1 \cap G_2$$

**Hypothesis 2:** We assume in the whole paper that for each node  $v \in K$  there exists a path in  $G$  that joins  $v$  with some vertex  $k_i \in K_1 \cap K_2$ .

These Hypothesis are illustrated in figure 3. In view of the first hypothesis, it is reasonable to assume the second one, otherwise  $R_K(G) = 0$  and there would be no necessity for any calculation. For notational convenience, the  $K$  subscript in  $R_K(G)$  will be omitted in the rest of the paper.

**Lemma 3.1.**  *$G$  is  $K$ -connected if and only if  $G$  is  $\{k_1, k_2, \dots, k_n\}$ -connected.*

*Proof:* The direct is trivial. Conversely, take a pair of nodes  $a$  and  $b$  in  $K$ . There are paths  $P_a$  y  $P_b$  connecting  $a$  and  $b$  with  $k_i$  y  $k_j$  respectively.

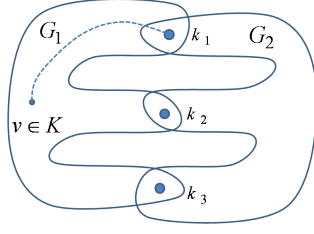


Figure 3: Hypothesis on the Graph

Because  $G$  is  $\{k_1, k_2, \dots, k_n\}$ -connected, there is a path  $P$  connecting  $k_i$  with  $k_j$ . The concatenation of the paths  $P_a$ ,  $P$  and  $P_b$  joins  $a$  with  $b$ .  $\square$

The previous lemma motivates the following definition.

**Definition 3.1.** Consider the equivalence relation:  $k_i \sim^l k_j$  if there is a path in  $G_l$  joining  $k_i$  with  $k_j$ ,  $l = 1, 2$ . The connectivity state of  $G_l$  is the partition of  $\{k_1, k_2, \dots, k_n\}$  given by

$$\mathcal{C}_l = \{k_1, k_2, \dots, k_n\} / \sim^l$$

Denote by  $Con$  the set of partitions of  $\{k_1, k_2, \dots, k_n\}$ . Figure 4 shows some useful notational and diagrammatical ways to represent a connectivity state.

**Definition 3.2.** For each connectivity state  $\mathcal{A}$  denote by  $G_l^{\mathcal{A}}$  the graph resulting of the identification of the nodes in  $\{k_1, k_2, \dots, k_n\}$  of  $G_l$  by the state  $\mathcal{A}$ ; i.e. given the graph  $G_l = (V, E)$  define  $G_l^{\mathcal{A}} = (V / \sim_l^{\mathcal{A}}, E^{\mathcal{A}})$  where

$$E^{\mathcal{A}} = \{ (\{\pi(a), \pi(b)\}, n) / (\{a, b\}, n) \in E \}$$

and  $\sim_l^{\mathcal{A}}$  is the equivalence relation in  $V$  generated by  $\mathcal{A}$  with the canonical surjection

$$\pi : V \rightarrow V / \sim_l^{\mathcal{A}}$$

such that  $\pi(a) = [a]_{\sim_l^{\mathcal{A}}}$ . The distinguished set of nodes is  $K_l^{\mathcal{A}} = \pi(K_l)$ .

**Definition 3.3.** The set of connectivity states of  $G$  is  $Con \times Con$  and  $(\mathcal{C}_1, \mathcal{C}_2)$  is the connectivity state of  $G$  where  $\mathcal{C}_l$  is the connectivity state of  $G_l$ ,  $l = 1, 2$ .

The rest of the section is devoted to the translation of the probabilistic problem given in the introduction into a purely combinatorial one.

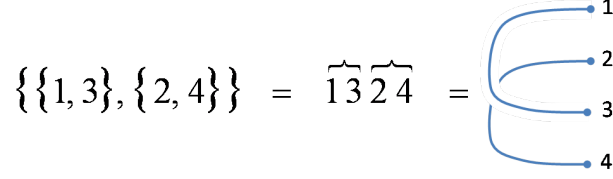


Figure 4: Connectivity State

**Definition 3.4.** Consider a stochastic graph  $G = (V, E, (p_e)_{e \in E})$  and define

$$\begin{aligned} P_l(\mathcal{C}) &= P(\mathcal{C} = \mathcal{C}_l) \\ P_G(\mathcal{A}, \mathcal{B}) &= P((\mathcal{A}, \mathcal{B}) = (\mathcal{C}_1, \mathcal{C}_2)) \end{aligned}$$

where  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  are connectivity states and  $l = 1, 2$ .

The connectivity states can be seen as a partition of the sample space of states of  $G_l$  where  $l = 1, 2$ , so

$$1 = \sum_{\mathcal{C} \in \text{Con}} P_l(\mathcal{C})$$

Because the random Bernoulli variables of the edges of  $G$  are independent, connectivity states of  $G_1$  are independent of those of  $G_2$  so  $\text{Con} \times \text{Con}$  is a partition of the sample space of states of  $G$  and

$$P_G(\mathcal{A}, \mathcal{B}) = P_1(\mathcal{A})P_2(\mathcal{B})$$

**Definition 3.5.** We say that a connectivity state  $(\mathcal{A}, \mathcal{B})$  of  $G$  is connected if

$$\{k_1, k_2, \dots, k_n\} / \sim = \{\{k_1, k_2, \dots, k_n\}\}$$

where  $\sim$  is the following equivalence relation in  $\{k_1, k_2, \dots, k_n\}$ :  $k_i \sim k_j$  if  $k_i \sim_{\mathcal{A}} k_j$  or  $k_i \sim_{\mathcal{B}} k_j$ .

Figures 5 and 6 show examples of connectivity states of  $G$ . This way we have the following formula for the  $K$ -reliability of  $G$ :

$$R(G) = \sum_{(\mathcal{A}, \mathcal{B}) \text{ connected}} P_G(\mathcal{A}, \mathcal{B}) = \sum_{(\mathcal{A}, \mathcal{B}) \text{ connected}} P_1(\mathcal{A})P_2(\mathcal{B})$$



Figure 5: Connected connectivity states of  $G$

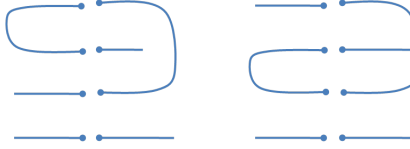


Figure 6: Non connected connectivity states of  $G$

also given in [Ro] (under a different notational scheme) where an algorithm for the reliability exact calculation is developed based on it. The above formula is not a factorization theorem. This formula suggests the following algebraic construction: Consider the  $\mathbb{Q}$ -vector space  $V$  generated by  $Con$  and the linear functional  $P_l : V \rightarrow \mathbb{R}$  which is the linear extension of the probability  $P_l$  (we are abusing of the notation and denoting the functional by the same name). This way we get the functional

$$P_1 \otimes P_2 : V \otimes V \rightarrow \mathbb{R}$$

and the following expression for the reliability:

$$R(G) = P_1 \otimes P_2 \left( \sum_{(\mathcal{A}, \mathcal{B}) \text{ connected}} \mathcal{A} \otimes \mathcal{B} \right)$$

**Definition 3.6.** Denote by  $R_{\mathcal{A}}(G_l)$  the  $K_l^{\mathcal{A}}$ -reliability of  $G_l^{\mathcal{A}}$

**Lemma 3.2.**  $R_{\mathcal{A}}(G_l) = \sum_{\mathcal{B} / (\mathcal{A}, \mathcal{B}) \text{ connected}} P_l(\mathcal{B})$

*Proof:* Observe that  $G_l^{\mathcal{A}}$  is  $K_l^{\mathcal{A}}$ -connected if and only if  $\mathcal{C}_l$  verifies that  $(\mathcal{A}, \mathcal{C}_l)$  is connected so

$$R_{\mathcal{A}}(G_l) = P(\mathcal{C}_l \in \{\mathcal{B} / (\mathcal{A}, \mathcal{B}) \text{ connected}\})$$



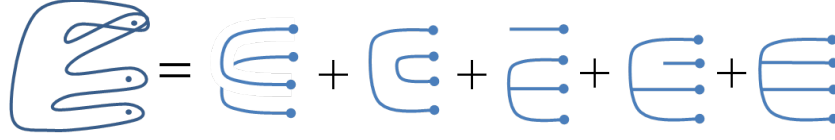


Figure 7

and because the set of connectivity states is a partition of the sample space of the states of  $G_l$ , we get the result.  $\square$

We can rewrite the above formula with the functional  $P_l$ :

$$R_{\mathcal{A}}(G_l) = P_l \left( \sum_{\mathcal{B} / (\mathcal{A}, \mathcal{B}) \text{ connected}} \mathcal{B} \right)$$

Then we have finally translated the original probabilistic problem in the following combinatorial one:

**Problem (Combinatorial Version):** Express the vector  $R \in V \otimes V$  given by

$$R = \sum_{(\mathcal{A}, \mathcal{B}) \text{ connected}} \mathcal{A} \otimes \mathcal{B}$$

in terms of the vectors  $R_{\mathcal{A}} \in V$  given by (see Figure 7)

$$R_{\mathcal{A}} = \sum_{\mathcal{B} / (\mathcal{A}, \mathcal{B}) \text{ connected}} \mathcal{B}$$

The relation between the combinatorial and the probabilistic problem is given by the functional  $P_1 \otimes P_2 : V \otimes V \rightarrow \mathbb{R}$ ; i.e. Acting on the combinatorial expression by  $P_1 \otimes P_2$  gives the probabilistic one.

#### 4. The Factorization Theorem

Considering an ordering in the base  $Con$  we can write

$$R = \sum_{i,j=1}^m a_{ij} \mathcal{A}_i \otimes \mathcal{A}_j$$

where  $m$  is the cardinal of  $Con$  and  $A = (a_{ij})$  is the matrix given by  $a_{ij} = 1$  if  $(\mathcal{A}_i, \mathcal{A}_j)$  is connected and  $a_{ij} = 0$  if it is not. The matrix  $A$  will be called

the *connectivity matrix*. Consider the linear operator  $T : V \rightarrow V$  such that  $T(\mathcal{A}) = R_{\mathcal{A}}$ .

**Lemma 4.1.** *The connectivity matrix is symmetric and is the associated matrix of the operator  $T$  relative to the ordered base  $Con$ .*

*Proof:* By definition,  $A$  is symmetric. Relative to the base  $Con$  we have

$$[T(\mathcal{A}_j)]_{Con} = [R_{\mathcal{A}_j}]_{Con} = (a_{1j}, a_{2j}, \dots, a_{mj})$$

so

$$[T]_{Con} = (a_{ij})$$

□

The next proposition will be proved in the next section (Proposition 5.7).

**Proposition 4.2.** *The operator  $T$  has determinant*

$$\det(T) = \pm \prod_{\mathcal{A} \in Con} (m_{\mathcal{A}} - 1)!$$

where  $m_{\mathcal{A}}$  is the number of classes in the connectivity state  $\mathcal{A}$ . In particular,  $T$  is an automorphism of  $V$ ; i.e.  $T \in \text{Aut}_{\mathbb{Q}}(V)$ .

*Observation 4.1.* The operator  ${}_Z T : \mathbb{Z}\langle Con \rangle \rightarrow \mathbb{Z}\langle Con \rangle$  such that  $T(\mathcal{A}) = R_{\mathcal{A}}$  whose relation with the operator  $T$  is

$$T = \mathbb{Q} \otimes {}_Z T$$

is not an automorphism of the  $\mathbb{Z}$ -vector space generated by  $Con$  if  $n > 2$ . In fact, its cokernel is (Proposition 5.10)

$$\text{Coker}({}_Z T) \simeq \frac{\mathbb{Z}\langle Con \rangle}{\mathbb{Z}\langle R_{\mathcal{A}} / \mathcal{A} \in Con \rangle} \simeq \bigoplus_{\mathcal{A} \in Con} \mathbb{Z}_{(m_{\mathcal{A}}-1)!}$$

where  $m_{\mathcal{A}}$  is the number of classes in the connectivity state  $\mathcal{A}$ .

*Observation 4.2.* If we write the above proposition in terms of the connectivity matrix,

$$\det(A) = \pm \prod_{\mathcal{A} \in Con} (m_{\mathcal{A}} - 1)!$$

we see that the left hand side of the equality is in some sense topological (it is related to connectedness) while the right hand side is combinatorial. It is a beautiful result.

Finally we have the factorization theorem which solves the posed combinatorial problem. This is the main theorem of the paper.

**Theorem 4.3.** *Let  $(b_{ij}) = A^{-1}$  where  $A$  is the connectivity matrix. Then*

$$R = \sum_{i,j=1}^m b_{ij} R_{\mathcal{A}_i} \otimes R_{\mathcal{A}_j}$$

*and the above expression doesn't depend on the order of the base  $Con$ .*

*Proof:* By lemma 4.1, we have

$$\mathcal{A}_j = \sum b_{ij} R_{\mathcal{A}_i}$$

so

$$R = \sum_{h,i,j,k=1}^m a_{ij} b_{ki} b_{hj} R_{\mathcal{A}_k} \otimes R_{\mathcal{A}_h}$$

Because  $(a_{ij})$  and  $(b_{ij})$  are inverses

$$\sum_{i=1}^m b_{ki} a_{ij} = \delta_{kj}$$

and we get

$$R = \sum_{h,j,k=1}^m \delta_{kj} b_{hj} R_{\mathcal{A}_k} \otimes R_{\mathcal{A}_h} = \sum_{h,k=1}^m b_{hk} R_{\mathcal{A}_k} \otimes R_{\mathcal{A}_h}$$

Finally, because  $A$  is symmetric, its inverse is too so

$$R = \sum_{h,k=1}^m b_{kh} R_{\mathcal{A}_k} \otimes R_{\mathcal{A}_h}$$

We have to show that the expression above is independent of the order in the base  $Con$ . It is enough to make an intrinsic formulation not depending on any order of the base:

$$\begin{aligned} R &= \sum_{\mathcal{A} \in Con} \mathcal{A} \otimes R_{\mathcal{A}} \\ &= \sum_{\mathcal{A} \in Con} T^{-1}(T(\mathcal{A})) \otimes R_{\mathcal{A}} \\ &= \sum_{\mathcal{A} \in Con} T^{-1}(R_{\mathcal{A}}) \otimes R_{\mathcal{A}} \end{aligned}$$

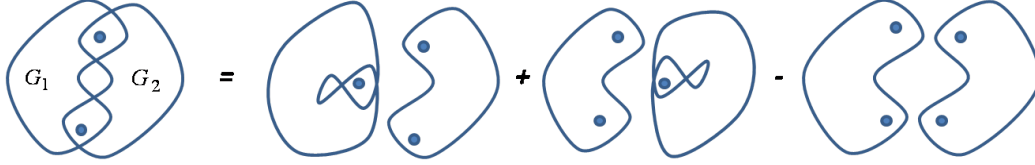


Figure 8

The above expression is the one we were looking for. Fixing an order in the base, the last expression reduces to the original one relative to this order.  $\square$

As it was mentioned in the previous section, we solved the posed probabilistic problem just applying the functional  $P_1 \otimes P_2$  on the last expression:

**Corollary 4.4.** *Under the hypothesis and notation of the previous theorem,*

$$R(G) = \sum_{i,j=1}^m b_{ij} R(G_1^{\mathcal{A}_i}) R(G_2^{\mathcal{A}_j})$$

The case  $n = 1$  is clear and reproduces the well known factorization respect to an articulation point. Let's see how the theorem works in a cutset of size two; i.e  $n = 2$ . Ordering the base  $Con$  by

$$Con = \{12, \widehat{12}\}$$

we get the connectivity matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

and its inverse

$$A^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$$

so

$$R = R_{1\widehat{2}} \otimes R_{12} + R_{12} \otimes R_{1\widehat{2}} - R_{12} \otimes R_{12}$$

Figure 8 shows this factorization. See the appendix for another proof of this result.

Let's see the case  $n = 3$ . Ordering the base  $Con$  by:

$$Con = \{123, 1\widehat{23}, \widehat{13}2, \widehat{12}3, \widehat{123}\}$$

we get the connectivity matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

and its inverse

$$A^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 & -1 & -1 & 2 \\ -1 & -1 & 1 & 1 & 0 \\ -1 & 1 & -1 & 1 & 0 \\ -1 & 1 & 1 & -1 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The following expression for the  $n = 3$  factorization is illustrated in Figure 9:

$$\begin{aligned} R &= R_{\widehat{123}} \otimes R_{123} + R_{123} \otimes R_{\widehat{123}} + \dots \\ &\dots + \frac{1}{2}(R_{123} \otimes R_{123} - R_{1\hat{2}3} \otimes R_{1\hat{2}3} - R_{1\hat{3}2} \otimes R_{1\hat{3}2} - R_{\hat{1}23} \otimes R_{\hat{1}23}) \\ &+ \frac{1}{2}(R_{1\hat{2}3} \otimes R_{1\hat{3}2} + R_{1\hat{2}3} \otimes R_{\hat{1}23} + R_{1\hat{3}2} \otimes R_{\hat{1}23} - R_{123} \otimes R_{1\hat{2}3} - R_{123} \otimes R_{1\hat{3}2} - R_{123} \otimes R_{\hat{1}23}) \\ &+ \frac{1}{2}(R_{1\hat{3}2} \otimes R_{1\hat{2}3} + R_{\hat{1}23} \otimes R_{1\hat{2}3} + R_{\hat{1}23} \otimes R_{1\hat{3}2} - R_{1\hat{2}3} \otimes R_{123} - R_{1\hat{3}2} \otimes R_{123} - R_{\hat{1}23} \otimes R_{123}) \end{aligned}$$

## 5. The Connectivity Matrix

### 5.1. Connectivity Matrix Determinant

In this section we will identify a connectivity state with its symmetry: i.e. there is a one to one correspondence

$$S_{\mathcal{A}} \rightarrow \mathcal{A}$$

between the connectivity states and the subgroups generated by transpositions of the permutation group  $S_n$  such that

$$\mathcal{A} = \{k_1, k_2, \dots k_n\}/S_{\mathcal{A}}$$

This means that the connectivity state  $\mathcal{A}$  is the set of orbits of  $\{k_1, k_2, \dots k_n\}$  under the action of the subgroup generated by transpositions  $S_{\mathcal{A}}$ .

$$\begin{aligned}
& \text{Diagram 1} = \text{Diagram 2} + \text{Diagram 3} \\
& + \frac{1}{2} ( \text{Diagram 4} - \text{Diagram 5} - \text{Diagram 6} - \text{Diagram 7} \\
& + \text{Diagram 8} + \text{Diagram 9} + \text{Diagram 10} - \text{Diagram 11} \\
& - \text{Diagram 12} - \text{Diagram 13} + \text{Diagram 14} + \text{Diagram 15} \\
& + \text{Diagram 16} - \text{Diagram 17} - \text{Diagram 18} - \text{Diagram 19} )
\end{aligned}$$

Figure 9

*Observation 5.1.* The subgroups must be generated by transpositions to get a maximal subgroup under the above condition and to get the inverse correspondence

$$\mathcal{A} \rightarrow S_{\mathcal{A}}$$

We could think of the connectivity states as representations of the subgroups generated by transpositions of the permutation group.

This identification shows that the connectivity states have a natural commutative monoid structure with unit

$$e = \{id\} \approx \{\{k_1\}, \{k_2\}, \dots, \{k_n\}\}$$

under the product

$$\mathcal{A} \cdot \mathcal{B} = \langle \mathcal{A} \cup \mathcal{B} \rangle$$

Because the product of subgroups generated by transpositions y generated by transpositions, this product is well defined. Observe that under the identification, *Con* has also a natural partial order given by inclusion. From now on we will denote by the same name *Con* the set of partitions of  $\{k_1, k_2, \dots, k_n\}$  and the set of subgroups generated by transpositions of the permutation group  $S_n$ .

This identification provides a useful and elegant way to characterize the connectivity state of  $G$ :  $(\mathcal{A}, \mathcal{B})$  is connected if and only if

$$\mathcal{A} \cdot \mathcal{B} = S_n$$

**Definition 5.1.** The permutation group  $S_n$  acts by conjugation in the connectivity states:

$$\sigma(\mathcal{A}) = \sigma \mathcal{A} \sigma^{-1}$$

where  $\mathcal{A} \in \text{Con}$ .

In particular we have conjugation classes in  $\text{Con}$ ; i.e. the orbits in  $\text{Con}$  under the action of  $S_n$ :  $\mathcal{A} \sim \mathcal{B}$  if there is a permutation  $\sigma$  such that  $\mathcal{A} = \sigma(\mathcal{B}) = \sigma \mathcal{B} \sigma^{-1}$ .

**Lemma 5.1.** Consider a permutation  $\sigma$  in  $S_n$ . Then  $\sigma : \text{Con} \rightarrow \text{Con}$  is unital monoid morphism; i.e.  $\sigma(\{id\}) = \{id\}$  and

$$\sigma(\mathcal{A} \cdot \mathcal{B}) = \sigma(\mathcal{A}) \cdot \sigma(\mathcal{B})$$

*Proof:* It is clear that  $\sigma(\{id\}) = \{id\}$ . Let's show that  $\sigma$  is a monoid morphism. See that  $\sigma(\mathcal{A} \cdot \mathcal{B})$  contains  $\sigma(\mathcal{A})$  and  $\sigma(\mathcal{B})$  so, by definition,

$$\sigma(\mathcal{A} \cdot \mathcal{B}) \supset \sigma(\mathcal{A}) \cdot \sigma(\mathcal{B})$$

See also that,  $\sigma^{-1}(\sigma(\mathcal{A}) \cdot \sigma(\mathcal{B}))$  contains  $\mathcal{A}$  and  $\mathcal{B}$  so, by definition,

$$\mathcal{A} \cdot \mathcal{B} \subset \sigma^{-1}(\sigma(\mathcal{A}) \cdot \sigma(\mathcal{B}))$$

Acting by  $\sigma$  we get

$$\sigma(\mathcal{A} \cdot \mathcal{B}) \subset \sigma(\mathcal{A}) \cdot \sigma(\mathcal{B})$$

□

This way, the  $\mathbb{Q}$ -vector space  $V$  generated by the commutative monoid  $\text{Con}$  is an associative and commutative  $\mathbb{Q}$ -algebra with unit. The unital monoid morphism  $\sigma$  extends linearly to a unital algebra morphism where  $\sigma$  is a permutation.

Consider the linear operator  $\pi : V \rightarrow V$  such that  $\pi(S_n) = S_n$  and for every connectivity state  $\mathcal{A}$  distinct from  $S_n$ ,

$$\pi(\mathcal{A}) = \prod_{\mathcal{B} / \mathcal{B} \not\sim \mathcal{A}} (\mathcal{A} - \mathcal{A} \cdot \mathcal{B})$$

**Lemma 5.2.** *Consider a connectivity state  $\mathcal{A}$ . The vector  $\pi(\mathcal{A})$  verifies the following properties:*

1.  $\mathcal{B} \cdot \pi(\mathcal{A}) = 0 \ \forall \mathcal{B} / \mathcal{B} \not\leq \mathcal{A}$
2.  $\mathcal{C} \cdot \pi(\mathcal{A}) = \pi(\mathcal{A}) \ \forall \mathcal{C} / \mathcal{C} \leq \mathcal{A}$
3.  $\pi(\sigma(\mathcal{A})) = \sigma(\pi(\mathcal{A})) \ \forall \sigma \in S_n$

*In particular,  $\mathcal{A} \cdot \pi(\mathcal{A}) = \pi(\mathcal{A})$  and  $\mathcal{B} \cdot \pi(\mathcal{A}) = 0$  for every connectivity state  $\mathcal{B}$  distinct and conjugated to  $\mathcal{A}$ .*

*Proof:*

1. The algebra is commutative and  $Con$  is a base of idempotents; i.e.  $\mathcal{A}^2 = \mathcal{A}$  for every connectivity state  $\mathcal{A}$ .
2. It is enough to see that  $\mathcal{C} \cdot \mathcal{A} = \mathcal{A}$  if  $\mathcal{C} \leq \mathcal{A}$ .
- 3.

$$\begin{aligned}
\sigma(\pi(\mathcal{A})) &= \sigma \left( \prod_{\mathcal{B} / \mathcal{B} \not\leq \mathcal{A}} (\mathcal{A} - \mathcal{A} \cdot \mathcal{B}) \right) \\
&= \prod_{\mathcal{B} / \mathcal{B} \not\leq \mathcal{A}} (\sigma(\mathcal{A}) - \sigma(\mathcal{A}) \cdot \sigma(\mathcal{B})) \\
&= \prod_{\sigma(\mathcal{B}) / \sigma(\mathcal{B}) \not\leq \sigma(\mathcal{A})} (\sigma(\mathcal{A}) - \sigma(\mathcal{A}) \cdot \sigma(\mathcal{B})) \\
&= \pi(\sigma(\mathcal{A}))
\end{aligned}$$

where we used in the last identity that  $\sigma$  is invertible.

□

**Definition 5.2.** For each connectivity state  $\mathcal{A}$  we define its connectivity number as the coefficient of  $S_n$  in the expression of  $\pi(\mathcal{A})$ ; i.e.

$$\pi(\mathcal{A}) = \dots + \alpha_{\mathcal{A}} S_n$$

See that, by the third item of lemma 5.2, the connectivity number is invariant under conjugation:

$$\alpha_{\mathcal{A}} = \alpha_{\sigma(\mathcal{A})}$$

for every permutation  $\sigma$ . The object of the following lemmas is to calculate the connectivity numbers and show that they are non zero.



**Definition 5.3.** Denote by  $\Gamma_n$  the graph with  $n$  nodes and one edge joining every pair of nodes. Denote by  $\Gamma_n^{\mathcal{A}}$  the resulting graph from the identification of the nodes  $\{1, 2, \dots, n\}$  in  $\Gamma_n$  by the classes of  $\mathcal{A}$ .

Let  $G$  be a stochastic graph and consider its reliability polynomial  $R(G)$ . Denote by  $mgr(R(G))$  the term of  $R(G)$  whose degree equals the edge number of  $G$ . See that if  $G$  has an irrelevant edge, then  $mgr(R(G)) = 0$  and in case  $mgr(R(G))$  is non zero, then this term equals the highest degree term of the polynomial.

**Lemma 5.3.** Consider a stochastic graph  $G$  with  $k$  edges between a pair of distinct nodes  $i$  and  $j$  of  $G$ . Consider the resulting graph  $\tilde{G}$  by deleting  $k - 1$  edges between the nodes  $i$  and  $j$  of  $G$ . Then,

$$mgr(R(G)) = (-p)^{k-1} mgr(R(\tilde{G}))$$

*Proof:* The result is clear for  $k = 1$ . Suppose there are  $k > 1$  edges between the nodes  $i$  and  $j$  of  $G$  and that the result holds for an amount less than or equal to  $k - 1$  of them. Consider an edge  $a$  between the nodes  $i$  and  $j$ . A simple factorization on the edge  $a$  gives

$$R(G) = p R(G \cdot a) + (1 - p) R(G - a)$$

where  $G \cdot a$  is the resulting graph by the contraction of  $a$  and  $G - a$  is the resulting graph by deleting  $a$ . See that the edge number of  $G \cdot a$  and  $G - a$  is the edge number of  $G$  minus one and because  $k > 1$ ,  $G \cdot a$  has irrelevant edges. This way,

$$mgr(R(G)) = (-p) mgr(R(G - a))$$

By the inductive hypothesis, we get the result.  $\square$

**Lemma 5.4.** The reliability polynomial of  $\Gamma_n$  is

$$R(\Gamma_n) = \pm(n - 1)! p^g + \dots$$

where the highest degree of the expression  $g$  equals the edge number of  $\Gamma_n$ :

$$g = \binom{n}{2}$$

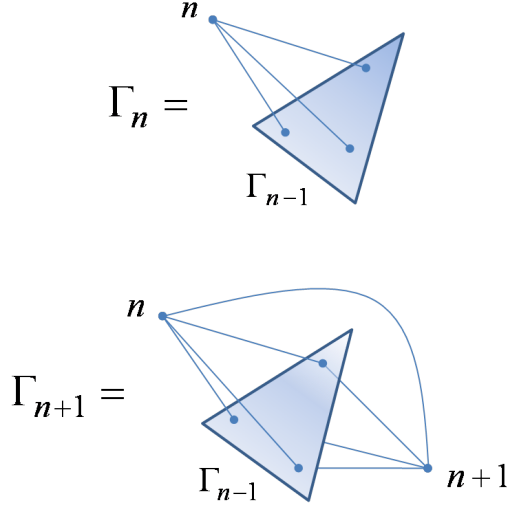


Figure 10: Relation between the graphs  $\Gamma$

*Proof:* In this proof we will make an abuse of notation identifying the reliability polynomial with its graph. We claim that

$$mgr(\Gamma_{n+1}) = (-1)^{n+1} n \, mgr(\Gamma_n) \, p^n$$

Because  $\Gamma_2 = p$  and  $\Gamma_1 = 1$  we have the result for the  $n = 1$  case. Suppose the claim is true for every natural less than or equal to  $n$ .

The Figure 10 shows the relation between the distinct graphs  $\Gamma$ . By a simple factorization on the edge joining the nodes  $n$  and  $n + 1$  of the graph  $\Gamma_{n+1}$  and the above lemma we have that

$$\Gamma_{n+1} = p(-p)^{n-1} \Gamma_n + \dots + (1 - p) H_{n+1}$$

where the dots denote terms whose degree is less than the edge number of  $\Gamma_{n+1}$  and the graph  $H_{n+1}$  results from deleting the edge joining the nodes  $n$  and  $n + 1$  of the graph  $\Gamma_{n+1}$ , see Figure 11.

By the inductive hypothesis,

$$mgr(\Gamma_n) = (-1)^n (n - 1) \, mgr(\Gamma_{n-1}) \, p^{n-1}$$

and the fact that the relation between the graphs  $\Gamma_n$  and  $\Gamma_{n-1}$  is the same as the one between  $H_{n+1}$  and  $\Gamma_n$  (see Figures 10 and 11), we have the following

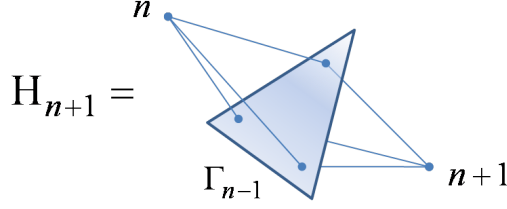


Figure 11: The graph  $H_{n+1}$

relation

$$mgr(H_{n+1}) = (-1)^n(n-1) mgr(\Gamma_n) p^{n-1}$$

Then, we have that

$$\Gamma_{n+1} = p(-p)^{n-1} \Gamma_n + \dots + (1-p)(-1)^n(n-1) \Gamma_n p^{n-1} + \dots = (-1)^{n+1}n \Gamma_n p^n + \dots$$

where the dots denote terms whose degree is less than the edge number of  $\Gamma_{n+1}$ . We conclude that

$$mgr(\Gamma_{n+1}) = (-1)^{n+1}n mgr(\Gamma_n) p^n$$

which proves the claim. This recursive relation shows that  $mgr(\Gamma_n)$  is non zero so it equals the highest degree term of  $\Gamma_n$ :

$$\Gamma_n = (-1)^{n+(n-1)+\dots+2}(n-1)!p^{(n-1)+\dots+1} + \dots$$

and this concludes the lemma.  $\square$

**Lemma 5.5.** *Consider a connectivity state  $\mathcal{A}$  with  $m$  classes  $a_1, a_2, \dots, a_m$ . Then*

$$R(\Gamma_n^{\mathcal{A}}) = \pm(m-1)! p^g + \dots$$

where the highest degree  $g$  of the expression equals the non irrelevant edge number of  $\Gamma_n^{\mathcal{A}}$ :

$$g = \sum_{i \neq j} a_i a_j$$

*Proof:* We will make the same abuse we did in the proof before identifying the reliability polynomial with its graph. The graph  $\Gamma_n^{\mathcal{A}}$  has  $m$  nodes (these

are the  $m$  classes of  $\mathcal{A}$ ,  $\binom{a_i}{2}$  irrelevant edges in each node  $i$  respectively and  $a_i a_j$  edges joining the nodes  $i$  and  $j$ .

Consider the graph  $\bar{\Gamma}_n^{\mathcal{A}}$  resulting from deleting all the irrelevant edges of the graph  $\Gamma_n^{\mathcal{A}}$ . This way  $\Gamma_n^{\mathcal{A}}$  and  $\bar{\Gamma}_n^{\mathcal{A}}$  have the same reliability polynomial. By the lemma 5.3 we have the following relation between the graphs  $\bar{\Gamma}_n^{\mathcal{A}}$  and  $\Gamma_m$ :

$$\begin{aligned} mgr(\bar{\Gamma}_n^{\mathcal{A}}) &= (-p)^{\sum_{i \neq j} (a_i a_j - 1)} mgr(\Gamma_m) \\ &= (-p)^{\sum_{i \neq j} (a_i a_j - 1)} (\pm(m-1)!) p^{\binom{m}{2}} \\ &= \pm(m-1)! p^{\sum_{i \neq j} a_i a_j} \end{aligned}$$

and this concludes the proof.  $\square$

**Lemma 5.6.** *Consider a connectivity state  $\mathcal{A}$  with  $m$  classes. Then,*

$$\alpha_{\mathcal{A}} = \pm(m-1)!$$

*Proof:* We claim that

$$\pi(\mathcal{A}) = \mathcal{A} \cdot \left( \prod_{\tau \text{ transp. / } \langle \tau \rangle \not\in \mathcal{A}} (e - \langle \tau \rangle) \right)$$

In effect, consider a connectivity state  $\mathcal{B}$  such that  $\mathcal{B} \not\in \mathcal{A}$ . There is a transposition  $\tau$  in  $\mathcal{B}$  not belonging to  $\mathcal{A}$ . Because  $\mathcal{A} \subset \mathcal{A} \cdot \mathcal{B}$  and  $\langle \tau \rangle \subset \mathcal{A} \cdot \mathcal{B}$  we have that  $\mathcal{A} \cdot \langle \tau \rangle \subset \mathcal{A} \cdot \mathcal{B}$  so

$$\mathcal{A} \cdot \mathcal{B} \cdot (\mathcal{A} - \mathcal{A} \cdot \langle \tau \rangle) = \mathcal{A} \cdot \mathcal{B} - \mathcal{A} \cdot \mathcal{B} = 0$$

This implies that

$$\mathcal{A} \cdot \mathcal{B} \cdot \left( \prod_{\tau \text{ transp. / } \langle \tau \rangle \not\in \mathcal{A}} (\mathcal{A} - \mathcal{A} \cdot \langle \tau \rangle) \right) = 0$$

so we have the following expression for the vector  $\pi(\mathcal{A})$ :

$$\pi(\mathcal{A}) = \prod_{\tau \text{ transp. / } \langle \tau \rangle \not\in \mathcal{A}} (\mathcal{A} - \mathcal{A} \cdot \langle \tau \rangle) = \mathcal{A} \cdot \left( \prod_{\tau \text{ transp. / } \langle \tau \rangle \not\in \mathcal{A}} (e - \langle \tau \rangle) \right)$$

which proves the claim. In particular, the last expression implies that

$$\alpha_{\mathcal{A}} = C_0 - C_1 + C_2 - C_3 + \dots$$

where  $C_i$  is the number of subsets  $F$  with cardinal  $i$  of the set of transpositions  $\{\tau \text{ transp.} / \langle \tau \rangle \not\subset \mathcal{A}\}$  such that  $\langle F \rangle \cdot \mathcal{A} = S_n$ .

Identifying the transposition  $(ij)$  with the edge joining the nodes  $i$  and  $j$  of  $\Gamma_n$ , it is clear that  $C_i$  is the pathsets number (operational states number) of  $\Gamma_n^{\mathcal{A}}$  with just  $i$  operational edges. This way we have that

$$\begin{aligned} R(\Gamma_n^{\mathcal{A}}) &= C_0(1-p)^g + C_1p(1-p)^{g-1} + C_2p^2(1-p)^{g-2} + \dots \\ &= (-1)^g(C_0 - C_1 + C_2 - C_3 + \dots)p^g \\ &= (-1)^g\alpha_{\mathcal{A}}p^g + \dots \end{aligned}$$

where  $g$  is the non irrelevant edge number of  $\Gamma_n^{\mathcal{A}}$  and the dots denote terms whose degree is less than  $g$ . By the above lemma this implies

$$\alpha_{\mathcal{A}} = \pm(m-1)!$$

□

Consider the connectivity state conjugation classes  $\{\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_k\}$ . The number of classes a connectivity state has is invariant under conjugation so we can define  $m_{\mathcal{O}_i}$  as the number of classes in some connectivity state in  $\mathcal{O}_i$ .

Identifying each level of the Hasse diagram (inclusion diagram) of the subgroups generated by transpositions of  $S_n$  and because conjugated states necessary belong to the same level of this diagram, we can order the connectivity states in  $Con$  in the following way: We order some conjugation class  $\mathcal{O}_i$  of the first level, then we order some other conjugation class  $\mathcal{O}_j$  of the same level and we continue the process until we have order all the conjugation classes of the first level. After that, we order the second level in the same way as we did in the first and so on until we have order all the connectivity states. The given order in  $Con$  verifies

$$\mathcal{A}_i < \mathcal{A}_j \Rightarrow i < j$$

We will call a coherent order an order in  $Con$  verifying the above property.

**Proposition 5.7.** *The determinant of the operator  $T$  such that  $T(\mathcal{A}) = R_{\mathcal{A}}$  is*

$$\det(T) = \pm \prod_{i=1}^k (m_{\mathcal{O}_i} - 1)!^{\#\mathcal{O}_i}$$

*Proof:* Choose a coherent order in the base  $Con$  and consider the connectivity matrix  $A$  relative to this order. Consider the matrix  $B$  associated to the operator  $\pi$  relative to the chosen coherent order in  $Con$ . Because of the identity proved in the last lemma

$$\pi(\mathcal{A}) = \mathcal{A} \cdot \left( \prod_{\tau \text{ transp. / } \langle \tau \rangle \not\in \mathcal{A}} (e - \langle \tau \rangle) \right)$$

we have that  $B$  is an inferior triangular matrix with ones in its diagonal,

$$B = \begin{pmatrix} 1 & 0 & \dots & 0 \\ * & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & 1 \end{pmatrix}$$

In particular,

$$\det(B) = 1$$

We can think about the expression of the vector  $\pi(\mathcal{A})$  in terms of the base  $Con$  as elementary row operations on the matrix  $A$  so, by lemma 5.2, we have the following identity:

$$B^t A = \begin{pmatrix} \alpha_{\mathcal{O}_1} I_{\# \mathcal{O}_1} & 0 & \dots & 0 \\ * & \alpha_{\mathcal{O}_2} I_{\# \mathcal{O}_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & \alpha_{\mathcal{O}_k} I_{\# \mathcal{O}_k} \end{pmatrix}$$

Because  $\det(B^t) = \det(B) = 1$  we conclude that

$$\det(A) = \prod_{i=1}^k \alpha_{\mathcal{O}_i} \# \mathcal{O}_i$$

and by the last lemma we finish the proof.  $\square$

As an example, let's see the  $n = 4$  case. The connectivity states conjugation classes are:

$$\begin{aligned}
\mathcal{O}_1 &= \{ 1234 \} \\
\mathcal{O}_2 &= \{ 12 \overbrace{34}, 13 \overbrace{24}, 23 \overbrace{14}, 1 \overbrace{23} 4, \overbrace{13} 24, \overbrace{12} 34 \} \\
\mathcal{O}_3 &= \{ \overbrace{14} \overbrace{23}, \overbrace{13} \overbrace{24}, \overbrace{12} \overbrace{34} \} \\
\mathcal{O}_4 &= \{ 1 \overbrace{234}, 2 \overbrace{134}, 3 \overbrace{124}, \overbrace{123} 4 \} \\
\mathcal{O}_5 &= \{ \overbrace{1234} \}
\end{aligned}$$

then we have that  $m_{\mathcal{O}_i}$  equals 4, 3, 2, 2, 1 respectively. By the above proposition the determinant of the connectivity matrix is:

$$\det(A) = \pm(4-1)!^1 (3-1)!^6 (2-1)!^3 (2-1)!^4 (1-1)!^1 = \pm 384$$

### 5.2. Connectivity Matrix Inverse

We can give another monoid structure on  $Con$ , defining the product as the intersection of the subgroups generated by transpositions where  $S_n$  is now the unit. In the same way as before, each permutation in  $S_n$  is a unital monoid morphism and its extension to a the  $\mathbb{Q}$ -algebra  $V$  (commutative and associative with unit, generated by the monoid  $Con$ ) is a unital algebra morphism.

Consider the linear operator  $\xi : V \rightarrow V$  such that  $\xi(\{id\}) = \{id\}$  and for every connectivity state  $\mathcal{A}$  distinct from  $\{id\}$  we have

$$\xi(\mathcal{A}) = \bigcap_{\mathcal{C} / \mathcal{C} \leq \mathcal{A}} (\mathcal{A} - \mathcal{C})$$

**Lemma 5.8.** *Consider a connectivity state  $\mathcal{A}$ . The vector  $\xi(\mathcal{A})$  verifies the following properties:*

1.  $\mathcal{B} \cap \xi(\mathcal{A}) = 0 \ \forall \mathcal{B} / \mathcal{B} \not\leq \mathcal{A}$
2.  $\mathcal{C} \cap \xi(\mathcal{A}) = \xi(\mathcal{A}) \ \forall \mathcal{C} / \mathcal{C} \geq \mathcal{A}$
3.  $\xi(\sigma(\mathcal{A})) = \sigma(\xi(\mathcal{A})) \ \forall \sigma \in S_n$

*Proof:* The same proof as in lemma 5.2. □

**Theorem 5.9.** *Consider a coherent order in  $Con$  and the matrices  $B$  and  $D$  associated to the operators  $\pi$  and  $\xi$  in the base  $Con$ . Consider also the matrix*

$$C = \begin{pmatrix} \alpha_{\mathcal{O}_1}^{-1} I_{\# \mathcal{O}_1} & 0 & \dots & 0 \\ 0 & \alpha_{\mathcal{O}_2}^{-1} I_{\# \mathcal{O}_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_{\mathcal{O}_k}^{-1} I_{\# \mathcal{O}_k} \end{pmatrix}$$

Then,

$$A^{-1} = B \ C \ D$$

*Proof:* Following the proof in 5.7, by lemma 5.2  $ij$  component of the matrix  $C \ B^t \ A$  is one if  $\mathcal{A}_j \leq \mathcal{A}_i$ ; i.e.  $\mathcal{A}_j \cap \mathcal{A}_i = \mathcal{A}_i$  and zero otherwise.

We can see the expression of the vector  $\xi(\mathcal{A})$  in terms of the base  $Con$  as elementary row operations on the matrix  $C \ B^t \ A$  and observing that this expression only has elements  $\mathcal{C}$  such that  $\mathcal{C} \preceq \mathcal{A}$ , by lemma 5.8 and the fact that  $\mathcal{B} \not\leq \mathcal{A}$  and  $\mathcal{C} \leq \mathcal{A}$  implies  $\mathcal{B} \not\leq \mathcal{C}$ , we have that

$$I = D^t \ C \ B^t \ A$$

Because  $A$  is symmetric, the transpose of the last expression shows that  $A$  has also a right inverse so  $A^{-1} = B \ C \ D$ .  $\square$

As an example consider the  $n = 3$  case. A coherent order on the base  $Con$  is

$$Con = \{123, 1 \overbrace{23}, \overbrace{13} 2, \overbrace{12} 3, \overbrace{123}\}$$

Let's Calculate the matrices  $B$ ,  $C$  y  $D$  of the previous theorem:

$$\begin{aligned} \pi(123) &= 123 - \overbrace{12} 3 - \overbrace{13} 2 - 1 \overbrace{23} + 2. \overbrace{123} \\ \pi(\overbrace{12} 3) &= \overbrace{12} 3 - \overbrace{123} \\ \pi(\overbrace{13} 2) &= \overbrace{13} 2 - \overbrace{123} \\ \pi(1 \overbrace{23}) &= 1 \overbrace{23} - \overbrace{123} \\ \pi(\overbrace{123}) &= \overbrace{123} \end{aligned}$$



and the matrix associated to  $\pi$  on the base  $Con$  is

$$B = \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ -1 & 0 & 1 & & \\ -1 & 0 & 0 & 1 & \\ 2 & -1 & -1 & -1 & 1 \end{pmatrix}$$

See that the entries of the last row are the respective connectivity numbers so

$$C = \begin{pmatrix} \frac{1}{2} & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & -1 & \\ & & & & 1 \end{pmatrix}$$

$$\begin{aligned} \xi(123) &= 123 \\ \xi(\widehat{12}3) &= \widehat{12}3 - 123 \\ \xi(\widehat{13}2) &= \widehat{13}2 - 123 \\ \xi(1\widehat{23}) &= 1\widehat{23} - 123 \\ \xi(\widehat{123}) &= \widehat{123} - \widehat{12}3 - \widehat{13}2 - 1\widehat{23} + 2 \cdot 123 \end{aligned}$$

and the matrix associated to  $\xi$  on the base  $Con$  is

$$D = \begin{pmatrix} 1 & -1 & -1 & -1 & 2 \\ & 1 & 0 & 0 & -1 \\ & & 1 & 0 & -1 \\ & & & 1 & -1 \\ & & & & 1 \end{pmatrix}$$

By the previous theorem, the connectivity matrix inverse is

$$A^{-1} = \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ -1 & 0 & 1 & & \\ -1 & 0 & 0 & 1 & \\ 2 & -1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & -1 & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 & -1 & 2 \\ & 1 & 0 & 0 & -1 \\ & & 1 & 0 & -1 \\ & & & 1 & -1 \\ & & & & 1 \end{pmatrix}$$

The permutation group  $S_3$  has the peculiarity that its minimal subgroups coincide with its maximal ones and this is the reason why the matrix  $D$  equals the transpose of  $B$  in this particular case. This phenomenon occurs only in this case. The inverse connectivity matrix calculation follows exactly the same steps as in the case before and it is realized through the algebra generated by the connectivity states monoid.

### 5.3. Invariant Factors

By the theorem 5.9 we can see the connectivity numbers unless sign as the invariant factors of the abelian torsion group:

$$\frac{\mathbb{Z}\langle Con \rangle}{\mathbb{Z}\langle R_{\mathcal{A}} / \mathcal{A} \in Con \rangle}$$

In effect, we have the following theorem:

**Theorem 5.10.**

$$\frac{\mathbb{Z}\langle Con \rangle}{\mathbb{Z}\langle R_{\mathcal{A}} / \mathcal{A} \in Con \rangle} \simeq \bigoplus_{i=1}^k \mathbb{Z}_{(m_{\mathcal{O}_i}-1)!}^{\# \mathcal{O}_i}$$

*Proof:* Following the proof of theorem 5.9, because  $B$  and  $D$  are triangular matrices with integer entries and ones in its diagonal,  $B$  and  $D$  are invertible in  $M_m(\mathbb{Z})$ ; i.e.

$$B, D \in GL_m(\mathbb{Z})$$

In particular  $\xi$  is an automorphism of  $\mathbb{Z}\langle Con \rangle$ :

$$\xi \in Aut(\mathbb{Z}\langle Con \rangle)$$

This way we have that

$$DA = \begin{pmatrix} \alpha_{\mathcal{O}_1} I_{\# \mathcal{O}_1} & 0 & \dots & 0 \\ 0 & \alpha_{\mathcal{O}_2} I_{\# \mathcal{O}_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_{\mathcal{O}_k} I_{\# \mathcal{O}_k} \end{pmatrix} B^{-1}$$

Consider the subgroup  $N$  of  $\mathbb{Z}\langle Con \rangle$  generated by the vectors  $\alpha_{\mathcal{A}} \mathcal{A}$  such that  $\mathcal{A}$  is a connectivity state. Because of lemma 4.1, the fact that  $B$  is invertible in  $M_m(\mathbb{Z})$  and the identity

$$\xi(R_{\mathcal{A}_j}) = \sum_{i=1}^m a_{ij} \xi(\mathcal{A}_i) = \sum_{i,k=1}^m d_{ki} a_{ij} \mathcal{A}_k = \sum_{i,k=1}^m \alpha_k \delta_{ki} e_{ij} \mathcal{A}_k = \sum_{i=1}^m e_{ij} (\alpha_i \mathcal{A}_i)$$

where  $(e_{ij}) = B^{-1}$  and  $(d_{ij}) = D$ , we conclude that  $N$  is isomorphic by  $\xi$  to the subgroup of  $\mathbb{Z}\langle \text{Con} \rangle$  generated by the vectors  $R_{\mathcal{A}}$ . We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z}\langle R_{\mathcal{A}} / \mathcal{A} \in \text{Con} \rangle & \longrightarrow & \mathbb{Z}\langle \text{Con} \rangle & \longrightarrow & \frac{\mathbb{Z}\langle \text{Con} \rangle}{\mathbb{Z}\langle R_{\mathcal{A}} / \mathcal{A} \in \text{Con} \rangle} \longrightarrow 0 \\
& & \downarrow \xi, \simeq & & \downarrow \xi, \simeq & & \downarrow \\
0 & \longrightarrow & N & \longrightarrow & \mathbb{Z}\langle \text{Con} \rangle & \longrightarrow & \bigoplus_{i=1}^k \mathbb{Z}_{(m_{\mathcal{O}_i}-1)!} \# \mathcal{O}_i \longrightarrow 0
\end{array}$$

where lemma 5.6 was used. By the five lemma (well known result in Homological Algebra, see [We]), we have the result.  $\square$

As an example, in the  $n = 3$  case we have

$$\frac{\mathbb{Z}\langle \text{Con} \rangle}{\mathbb{Z}\langle R_{\mathcal{A}} / \mathcal{A} \in \text{Con} \rangle} \simeq \mathbb{Z}_2$$

while in the  $n = 4$  case we have

$$\frac{\mathbb{Z}\langle \text{Con} \rangle}{\mathbb{Z}\langle R_{\mathcal{A}} / \mathcal{A} \in \text{Con} \rangle} \simeq \mathbb{Z}_6 \oplus \mathbb{Z}_2^6 \simeq \mathbb{Z}_3 \oplus \mathbb{Z}_2^7$$

In contrast to the  $n = 3, 4$  cases, the invariant factors in the  $n > 4$  cases are not just the prime decomposition of the connectivity matrix determinant. For example, consider the  $n = 5$  case:

$\mathcal{O}$	$\#\mathcal{O}$	$m_{\mathcal{O}}$
[12345]	1	5
[ $\overbrace{12}$ 345]	10	4
[ $\overbrace{12}$ $\overbrace{34}$ 5]	15	3
[ $\overbrace{123}$ 45]	10	3
[ $\overbrace{1234}$ 5]	5	2
[ $\overbrace{123}$ $\overbrace{45}$ ]	10	2
[ $\overbrace{12345}$ ]	1	1

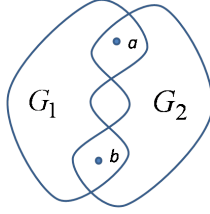
and we have that

$$\frac{\mathbb{Z}\langle \text{Con} \rangle}{\mathbb{Z}\langle R_{\mathcal{A}} / \mathcal{A} \in \text{Con} \rangle} \simeq \mathbb{Z}_{4!} \oplus \mathbb{Z}_{3!}^{10} \oplus \mathbb{Z}_2^{15} \oplus \mathbb{Z}_2^{10} \simeq \mathbb{Z}_8 \oplus \mathbb{Z}_3^{11} \oplus \mathbb{Z}_2^{35}$$

## Appendix A. Another Proof of the $n=2$ case

In this section, connected means edge-connected.

**Lemma Appendix A.1.** *Consider a graph  $G$  and subgraphs  $G_1$  and  $G_2$  with distinguished sets of nodes  $K_1$  and  $K_2$  respectively such that  $G = G_1 \cup G_2$  and  $\{a, b\} = K_1 \cap K_2 = G_1 \cap G_2$  (see the next Figure). Define  $K = K_1 \cup K_2$ . If  $G$  is  $K$ -connected, then  $\hat{G}_1$  is  $\hat{K}_1$ -connected and  $\hat{G}_2$  is  $\hat{K}_2$ -connected, where  $\hat{G}_i$  and  $\hat{K}_i$  is the graph and set resulting from the identification of the nodes  $a$  and  $b$  in the graph and set  $G_i$  and  $K_i$  respectively.*



*Proof:* Consider  $k$  in  $K_1$ . There is a path in  $G$  connecting  $k$  with  $a$ . Then, there is a subpath in  $G_1$  connecting  $k$  with  $a$  or  $b$ ; i.e., there is a path in  $\hat{G}_1$  connecting  $k$  with  $a = b$ . This is true for every  $k$  in  $\hat{K}_1$  so  $\hat{G}_1$  is  $\hat{K}_1$ -connected. In the same way as before,  $\hat{G}_2$  is  $\hat{K}_2$ -connected.  $\square$

**Lemma Appendix A.2.** *Under the hypothesis of the previous lemma,  $G$  is  $K$ -connected if and only if one of the following items holds:*

- i.  $G_1$  is  $K_1$ -connected and  $\hat{G}_2$  is  $\hat{K}_2$ -connected.*
- ii.  $G_2$  is  $K_2$ -connected and  $\hat{G}_1$  is  $\hat{K}_1$ -connected.*

*Proof:* By the previous lemma  $\hat{G}_1$  is  $\hat{K}_1$ -connected and  $\hat{G}_2$  is  $\hat{K}_2$ -connected. Suppose that  $G_2$  is not  $K_2$ -connected; i.e., there are nodes  $u, v$  in  $K_2$  such that there is no path in  $G_2$  connecting them.

We claim that  $G_1$  is  $K_1$ -connected. Suppose that it is not; i.e., there are nodes  $x, y$  in  $K_1$  such that there is no path in  $G_1$  connecting them. However,  $G$  is  $K$ -connected through paths  $\gamma_1$  and  $\gamma_2$  in  $G$  connecting  $x$  with  $u$  and  $y$  with  $v$  respectively. Because  $\gamma_1$  and  $\gamma_2$  contain the nodes  $a$  or  $b$  (but not the same node), we conclude that  $a$  or  $b$  are not connected in  $G$ . In effect, if they were connected by a path in  $G$ , then it would exist a subpath connecting them in  $G_1$  or  $G_2$  so  $x$  and  $y$  or  $u$  and  $v$  would be connected. This is absurd.

because  $G$  is  $K$ -connected and the nodes  $a$  and  $b$  are in  $K$ . We conclude that  $G_1$  is  $K_1$ -connected.

Conversely, suppose without loss of generality that  $G_1$  is  $K_1$ -connected and  $\hat{G}_2$  is  $\hat{K}_2$ -connected. We claim that  $G$  is  $K$ -connected. In effect, consider  $k_1$  and  $k_2$  in  $K$ . If  $k_1$  and  $k_2$  are in  $K_1$ , then there is a path in  $G_1$  connecting  $k_1$  with  $k_2$  and because  $G_1 \subset G$  this path is also in  $G$ . If  $k_1$  is in  $K_1$  and  $k_2$  is in  $K_2$ , then there is a path in  $G_2$  connecting  $k_2$  with  $a$  or  $b$ . Suppose without loss of generality that  $k_2$  is connected with  $a$ . There is a path in  $G_1$  connecting  $a$  with  $k_1$ . The concatenation of these paths connects  $k_1$  with  $k_2$  in  $G$ . The argument for the case  $k_1$  in  $K_2$  and  $k_2$  in  $K_1$  is the same. Finally, if  $k_1$  and  $k_2$  are in  $K_2$ , then there are paths in  $G_2$  connecting  $k_1$  and  $k_2$  with  $a$  or  $b$ . If these paths connect  $k_1$  and  $k_2$  with the same point, then  $k_1$  and  $k_2$  are connected in  $G_2 \subset G$ , otherwise there is a path in  $G_1$  connecting  $a$  with  $b$  and the concatenation of these three paths connects  $k_1$  with  $k_2$  in  $G$ .  $\square$

As a corollary we have the factorization formula in the  $n = 2$  case (see figure 8).

**Theorem Appendix A.3.** *Under the hypothesis of lemma Appendix A.1 we have that*

$$R_K(G) = R_{K_1}(G_1)R_{\hat{K}_2}(\hat{G}_2) + R_{\hat{K}_1}(\hat{G}_1)R_{K_2}(G_2) - R_{K_1}(G_1)R_{K_2}(G_2)$$

*Proof:* By the previous lemma Appendix A.2, the set of  $K$ -PathSets is the union of the sets of states  $C_1$  y  $C_2$  which verify the items  $i$  and  $ii$  of the lemma. The intersection of these sets is the set of states verifying that  $G_1$  is  $K_1$ -connected and  $G_2$  is  $K_2$ -connected. From the identity

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

and the independence follows the result.  $\square$

As a corollary, taking  $a = b$  in the previous formula we get the well known factorization respect to an articulation node.

## Appendix B. The n=4 case

For the case  $n = 4$ , ordering the base  $Con$  by:

$$Con = \{1234, 12 \overbrace{34}, 13 \overbrace{24}, 23 \overbrace{14}, 1 \overbrace{23} 4, \overbrace{13} 24, \overbrace{12} 34, \overbrace{14} \overbrace{23}, \dots \\ \dots \overbrace{13} \overbrace{24}, \overbrace{12} \overbrace{34}, 1 \overbrace{234}, 2 \overbrace{134}, 3 \overbrace{124}, \overbrace{123} 4, \overbrace{1234}\}$$

we get the connectivity matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

and its inverse

$$A^{-1} = \frac{1}{6} \begin{pmatrix} -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -2 & -2 & -2 & -2 & 6 \\ 1 & 2 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & -2 & -1 & -1 & 2 & 2 & 0 \\ 1 & -1 & 2 & -1 & -1 & -1 & -1 & 1 & -2 & 1 & -1 & 2 & -1 & 2 & 0 \\ 1 & -1 & -1 & 2 & -1 & -1 & -1 & -2 & 1 & 1 & 2 & -1 & -1 & 2 & 0 \\ 1 & -1 & -1 & -1 & 2 & -1 & -1 & -2 & 1 & 1 & -1 & 2 & 2 & -1 & 0 \\ 1 & -1 & -1 & -1 & -1 & 2 & -1 & 1 & -2 & 1 & 2 & -1 & 2 & -1 & 0 \\ 1 & -1 & -1 & -1 & -1 & -1 & 2 & 1 & 1 & -2 & 2 & 2 & -1 & -1 & 0 \\ -1 & 1 & 1 & -2 & -2 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 0 \\ -1 & 1 & -2 & 1 & 1 & -2 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 0 \\ -1 & -2 & 1 & 1 & 1 & 1 & -2 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 0 \\ -2 & -1 & -1 & 2 & -1 & 2 & 2 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 \\ -2 & -1 & 2 & -1 & 2 & -1 & 2 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 \\ -2 & 2 & -1 & -1 & 2 & 2 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 \\ -2 & 2 & 2 & 2 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R = R_{\widehat{1234}} \otimes R_{1234} + R_{1234} \otimes R_{\widehat{1234}} + \dots$$

### Appendix C. Factorization of the Random Cluster Model in the non Cluster Limit

In Statistical Mechanics, the random cluster model [Gri] partition function of a stochastic edge connected graph  $G = (V, E, (p_e)_{e \in E})$  is given by

$$Z(q, G) = \sum_{\omega \in \Omega} \prod_{e \in E} q^{k(\omega)} p_e^{\omega(e)} (1 - p_e)^{1 - \omega(e)}$$

where  $\Omega = \{0, 1\}^E$  is the set of states and  $k(\omega)$  is the number of connected components of the state  $\omega$ . The parameter  $q$  gives weight based on the number of closed sets of connected vertices including isolated ones; i.e. clusters. The  $q = 1$  case is the percolation model and the  $q > 1$  case prefers more clusters while the  $q < 1$  case prefers fewer clusters. The non cluster limit is the  $q \approx 0$  case. It is easy to see that a factorization of  $G$  as the one described in the paper is realized as the product of partition functions in percolation theory:

$$Z(1, G) = Z(1, G_1) Z(1, G_2)$$

and this is the only known factorization in the cluster random model. Because of the following relation between the random cluster model and the all terminal reliability:

$$Z(q, G) = q R_{All}(G) + O_2(q)$$

we have the following factorization in the non cluster limit (see Theorem 4.3):

$$\frac{\partial Z(q, G)}{\partial q} \Big|_{q=0} = \sum_{i,j=1}^m b_{ij} \frac{\partial Z(q, G_1^{A_i})}{\partial q} \Big|_{q=0} \frac{\partial Z(q, G_2^{A_j})}{\partial q} \Big|_{q=0}$$

where  $b$  is the connectivity matrix inverse.

### References

- [Bi] N.L.Biggs, *Algebraic Graph Theory*, Cambridge, Cambridge University Press, 1993.
- [Co] C.J.Colbourn, *The Combinatorics of Network Reliability*, New York, Oxford University Press, 1987.



- [Ga] J.P.Gadani, *System effectiveness evaluation using star and delta transformations*, IEEE Trans.Reliability, R-30, No1, 41-47, 1981.
- [Gri] G.Grimmet, *The Random-Cluster Model*, Springer, Berlin, 2006.
- [Mo] F.Moskovitz and R.A.D.Center, *The analysis of redundancy networks*, Rome Air Development Center, Air Research and Development Center, United States Air Force, 1958.
- [Ro] A.Rosenthal, *Computing the Reliability of Complex Networks*, Siam J.Applied Math., 32, No2, 384-393.
- [Rot] J.J.Rotman, *Advanced Modern Algebra*, Prentice Hall, 2nd printing, 2003.
- [SC] A.Satyanarayana, M.Chang, *Network Reliability and the Factoring Theorem*, Networks, 13 (1983), 107-120.
- [We] C.A.Weibel, *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics, vol.38, Cambridge University Press, 1994.
- [Wo] R.K.Wood, *A factoring algorithm using polygon-to-chain reductions for computing k-terminal network reliability*, Networks, 15 (1985), 173-190.